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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. The series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) Euclidean constructions: (2) the geometry of the fixed compass: (3) certain topics related to constructions with straightedge and compasses: and (4) unorthodox ways to trisect a line segment. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

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PREFACE

When we use the phrase "geometric drawing," the implication is that there are no restrictions as to what instruments may be used. The expression "geometric construction" generally implies, either tacitly or explicitly, that the "drawing" is to be executed by the use of certain instruments and no others. The reader should also be reminded of the fact that a geometric figure is an abstraction—an "ideal" configuration—whereas the actual drawing which we produce is a physical thing, merely an approximation to the ideal configuration.

When we refer to a *ruler* and a *compass* we have in mind an "ideal ruler" and an "ideal compass" with which to draw straight lines and circles "exactly," in the sense that the thickness of the pencil marks and other imperfections due to the mechanics of drafting are simply ignored. A ruler may be marked or unmarked; the unmarked ruler is generally called a straightedge. A compass may "collapse" when lifted off the paper, or it may remain "open" or set at a fixed radius. A compass that does not collapse is more properly called a pair of dividers; its purpose is to transfer distances, not to draw circles.

The study of geometric constructions and geometrography has many facets. Thus all the constructions of elementary geometry can be carried out by using only a straightedge and collapsible compass. This was the technique that the Greek geometers used. But it is interesting to note that these two instruments are not entirely necessary, for there are many constructions in which only one or the other is required, not both. Indeed, in modern times it was found that the straightedge can be dispensed with altogether, and that all constructions that are possible with straightedge and compass can be made with the compass alone, assuming that a line is considered as having been "constructed" as soon as two of its points have been determined. This technique, due first to Mohr and later to Mascheroni, is sometimes called "compass geometry."

Other techniques are also possible. Thus Jacob Steiner (c. 1830) was one of the first to recognize the fact that any point which can be constructed with straightedge and compasses can be constructed with the straightedge alone, provided that a fixed circle and its center are given in the plane of the construction. Another interesting technique is that of Poncelet (c. 1820), who showed that every point constructed with str·ightedge and compasses can be constructed with a two-edged ruler al· ıe, such as the carpenter's square.

The essays which follow give an account of some of these techniques. For a discussion of Mascheroni constructions, the reader may consult another in this series of reprint pamphlets.

The reader should realize that these four articles are only representative of many articles, monographs, and books dealing with various aspects of geometric constructions. A considerable number of such sources are given in the bibliography following Hess's article, as well as in the list of *"Selected References for Further Reading and Study"* which appears at the end of the booklet. In this connection, may we su·.gest that the Hilda Hudson monographs constitute an especially valuable source of further information.

—William Schaaf

CONTENTS

ACKNOWLEDGMENTS

The School Mathematics Study Group takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reproduced in this manner. At the time that the late Professor Yates' article was written he was a Colonel attached to the United States Military Academy at West Point, teaching mathematics; subsequently he was chairman of the Departments of Mathematics at William and Mary at Williamsburg and at the University of South Florida at Tampa, respectively. Arthur Hallerberg was associated with Illinois College, Jacksonville, Illinois, at the time his paper was first published. Adrien Hess is an occasional contributor to mathematical journals. Charles W. Trigg, of the Los Angeles City College, Los Angeles, California, is a frequent contributor of articles for periodicals and a veteran contributor to the Problems and Solutions departments of these periodicals.

The School Mathematics Study Group is also pleased to express its sincere appreciation to the several editors and publishers who have been kind enough to permit these articles to be published in this manner, namely:

THE MATHEMATICS TEACHER:

(1) Arthur E. Hallerberg, *"The Geometry of the Fixed Compass,"* vol. 52, p. 230-244; April 1959.

(2) Robert C. Yates, *"Euclidean Constructions,"* vol. 47. p. 231-233; April 1954.

SCHOOL SCIENCE AND MATHEMATICS:

(1) Charles W. Trigg, *"Unorthodox Ways to Trisect a Line Segment,"* vol. 54, p. 525-528; 1954.

MATHEMATICS MAGAZINE:

(1) Adrien L. Hess, *"Certain Topics Related to Constructions with Straightedge and Compasses,"* vol. 29, p. 217-221; 1956.

FOREWORD

In this first essay the author very lucidly points out the actual limitations which the Greek geometers imposed upon their instruments when discussing the "construction" of geometric figures. He also suggests that the true Euclidean "rules of the game" are not always explicitly observed in modern texts on geometry.

The next essay continues this line of thought by discussing the geometry of the "fixed-compass." Although the "collapsible-compass" geometry was a bequest of the Greeks, the high school geometry of the present is essentially a rigid-compass geometry. The present article gives an excellent account of constructions that are possible with a straightedge and a *compass restricted to one and the same opening throughout the entire construction*, a technique referred to as the "geometry of the fixed-compass."

Although the term is not widely use, *geometrography* refers to various aspects of geometric constructions, including a consideration of constructions with limited instruments; constructions with obstructions in the plane; other instruments beside straightedge and compass; three-dimensional geometry; descriptive geometry; etc. The third essay gives the reader the flavor of geometrography, touching also upon paper-folding and match-stick geometry.

In the final essay we are given further insight into the general approach to methods of geometrography.

Euclidean Constructions

Robert C. Yates

In the spirit of the old-time revival and the spring tonic, I feel it periodically necessary to "reaffirm the faith" and refresh myself in the fundamental constructions of Euclidean plane geometry. It seems always such a satisfying experience that I wish to share it. My refreshment takes the following form.

The geometry of Plato and Euclid is built upon assumptions regarding coexistence of elements. Chief among these are the ones of incidence:

1. A straight line of indefinite length is determined by two points;
2. A circle is determined by two points, one of which is its center.

Other assumptions permit identification of a point upon a line, the point upon two lines, etc.

Following the Euclidean tradition, I have found it helpful to imagine, construct, and use a "straightedge" whose copy will visualize the line and I have also invented the "compasses" to produce a physical circle.

In accordance with these assumptions I *may:*

I. establish a straight line only upon two points,

II. draw a circle with given center only if also given a point through which it passes;

and I *may not:*

III. have measuring marks upon the straightedge,

IV. rigidly attach two straightedges, parallel or not,

carry lengths with the compasses (i.e., use the compasses as dividers),

VI. have a circular disc with measuring marks upon its periphery (i.e., a protractor. An unmarked circular disc together with straightedge would be permissible but inconvenient),

VII. attach a circular disc to a straightedge,

VIII. compound the compasses (i.e., use an instrument to draw concentric circles at the same time; use a linkage).

With these privileges and restrictions in mind, I shall look closely now at some fundamental and important constructions.

1. ***Bisect the given angle θ.*** **Figure 1.**

Select (identify) a point P on one side. Draw circle $O(P)$[1] meeting the other side in Q. Draw $Q(P)$ and $P(Q)$ which meet in X. Draw the bisector through X. Modern methods violate V and VI.

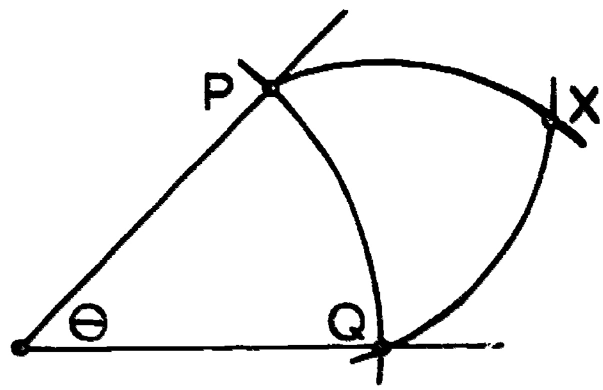


FIGURE 1

2. *Erect the perpendicular to a line k at a point P.* Figure 2.

Select Q on k and draw $P(Q)$ meeting k again in R. Then $Q(R)$ and $R(Q)$ meet in X, and PX is the desired perpendicular.

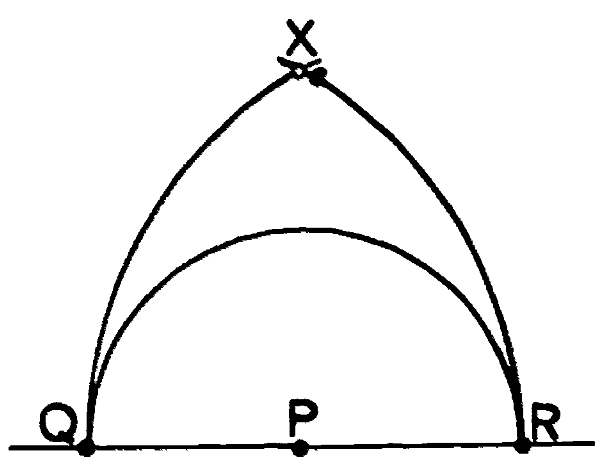


FIGURE 2

3. *Draw the parallel to a line k through a point P.* Figure 3.

Select a point Q on k. Draw $P(Q)$. Draw $Q(P)$ meeting k in S. Draw $S(Q)$ meeting $P(Q)$ in X which, with P, determines the parallel to k. I notice that only three circles (all with the same radius) need be drawn to locate the final point X. This is a measure of economy that I compare with modern textbook constructions.

[1] This notation is for the circle through P with center O.

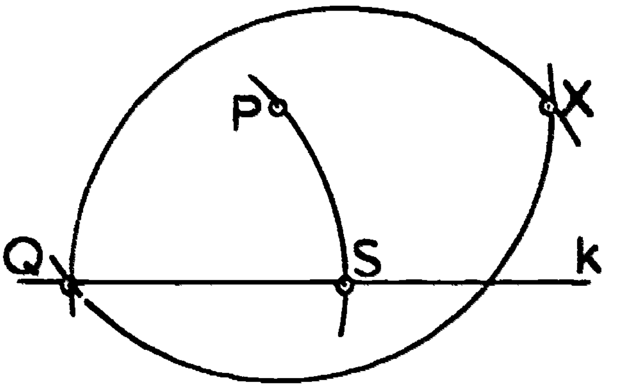


FIGURE 3

4. *Draw the circle with center C and radius AB* $(=r)$. Figure 4.

Since the radius is not given in position I must establish a point P such that $CP=AB=r$. Thus draw $C(A)$ and $A(C)$ meeting in Q. Then $CQ=AQ$. Draw $A(B)$ meeting AQ in R. Then $AR=r$. Let $RQ=x$. Draw $Q(R)$ meeting CQ in P. Then $PQ=x$ and, since $CQ=AQ=r+x$, then $CP=r$. The circle $C(P)$ is the one desired. Modern methods violate V.

I have achieved here a real advantage. In effect, I have won the privilege of incorporating the dividers into the compasses and can henceforth with clear conscience use the noncollapsible "modern" compasses.

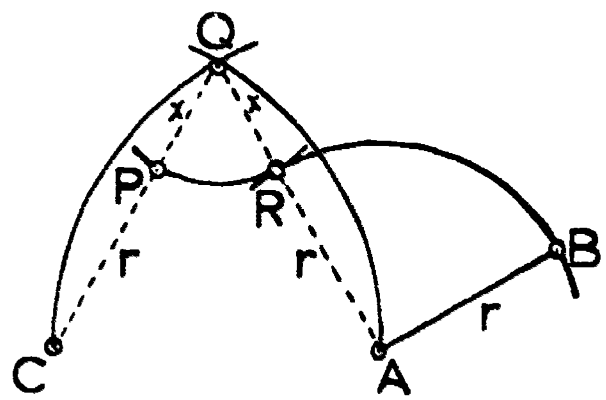


FIGURE 4

I should clearly understand that a great many instruments currently in vogue in modern geometry courses are very sophisticated tools. The privilege of using a marked straightedge (a ruler) is simply and legitimately acquired through the compasses. The crime is that I use it with disregard of proprieties often without thinking. The straightedge with but two marks on it is capable of solving problems which may be represented by quartic equations whose coefficients are given lengths. For example, I may use it to trisect any angle AOB—a cubic problem—as follows: Let the ruler have marks P, Q upon it distant $2a$ apart. Establish $OC=a$ on OB and from C draw a parallel CX and a perpendicular CY to OA. Place the straightedge through O and move P on CY. When Q falls on CX the angle is trisected. For, let $AOQ=\theta$ and call M the mid-point of $PQ=2a$. Then $\measuredangle OQC=\theta$ (parallels and a transversal) and $\measuredangle MCQ=\theta$ since $MC=MQ=a$. Furthermore, $\measuredangle OMC=2\theta$ and, since $OC=CM=a$, $\measuredangle MOC=2\theta$. *QED.*

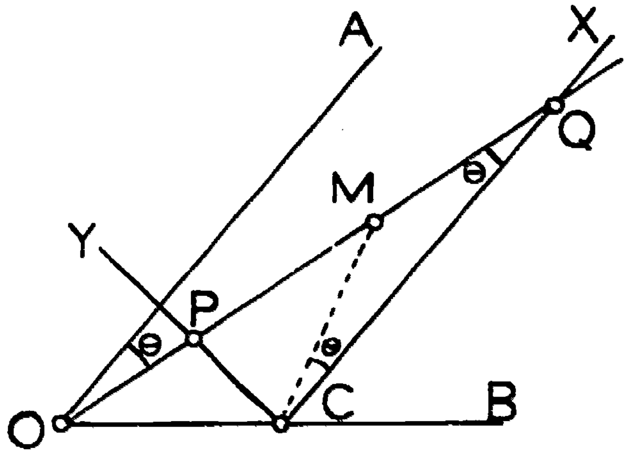


FIGURE 5

Now to disclose and face the facts involved in this trisection. First of all, I understand that Euclidean plane constructions are representable as nothing more than quadratics. Here, however, I notice the passage of Q. The straightedge is kept upon O, and P moved along CY. The path of Q is a *Conchoid*, a curve of *fourth* degree in rectangular coordinates. I have thus acquired in non-classical fashion the intersection of this curve and the line CX. (To move P and Q upon CY and CX and stop when the straightedge falls upon O is to exchange equivalent miseries. The segment PQ is tangent to an *Astroid*, another curve of fourth degree.)

The protractor has no place in our course in geometry. I find the angle *one degree* clearly marked on an expensive model in my possession. But not even the instrument company can construct one degree. As an instrument of practical measurement it has its use, to be sure, and ranks with the carpenter's square as a tool for the jobber. But in the hands of a student of that finest of arts called geometry, it serves only to contaminate and confuse.

And now that the experiences of my revival are over I have a clearer picture of the nature of Euclidean constructions. Perhaps this clarity will be reflected by my students in their knowledge, understanding, and appreciation of the structure of their geometry.

The Geometry of the Fixed-compass

Arthur E. Hallerberg

Topics in mathematics which clarify mathematical meanings, which develop mathematical appreciations, and which furnish opportunities for the student to "discover" mathematical ideas provide stimulating material for both teacher and student. If such a topic has an interesting or significant historical development as well, it has an even greater contribution to offer.[1]

This paper is concerned with such a topic: the question of how various geometrical constructions can be performed by using a straightedge and a compass[2] restricted to one and the same opening throughout the entire construction. We shall find it convenient to refer to this more briefly as "the geometry of the fixed-compass."

Introducing the Geometry of the Fixed-compass

Let us suppose that a geometry class which has studied the elementary geometrical constructions with ordinary compass and straightedge is asked to isolate a significant similarity in the constructions for bisecting a given line segment and for bisecting a given angle. It is possible that some student will notice that a single opening of the compass can be used throughout each construction, and, in fact, that the width of this opening may be arbitrary. (Students may at first feel that "too small" an opening makes the solution impossible, but a little thought will soon indicate that this is no real hardship — see Figure 1, where segment AB is bisected with a compass opening less than half of AB.) The students are then challenged to try to discover if there are other basic constructions which can be carried out using a "rusty compass," in which the opening between the legs (the radius of the arcs or circles drawn) is never changed. The use of the unmarked straightedge (ruler) in addition to the fixed-compass is of course assumed throughout all of this discussion.

[1]See P. S. Jones, "The History of Mathematics as a Teaching Tool," THE MATHEMATICS TEACHER, L (January 1957), 59–64; Herta T. and Arthur H. Freitag, "Using the History of Mathematics in Teaching on the Secondary School Level," THE MATHEMATICS TEACHER, L (March 1957), 220–224.

[2]"Usage favors the singular, compass, to refer to a single instrument and the plural, compasses, to refer to more than one, although pair of compasses, referring to the single instrument with its pair of legs, is used." *The Encyclopedia Americana*, 1957 edition, VII, 427.

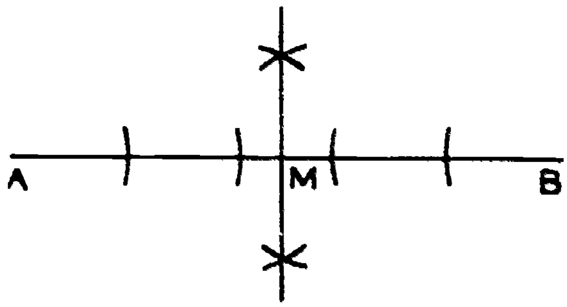


FIGURE 1

It may be worthwhile to suggest such problems to students as the construction of perpendiculars from points on and off a given line; the drawing of the parallel to a given line through a given point; and the division of a given segment into any number of equal parts. More challenging problems would be the transfer of a given angle; the drawing of a triangle, given the lengths of the three sides (each unequal to the given compass opening); and the inscribing of a regular pentagon in a given circle.

Some of the better students in a class should be successful in finding solutions for some of the basic constructions under these limitations. A pooling of ideas and particularly an analysis of the use of intermediate constructions, which can be combined into more complicated constructions, suggest the value of developing something of a systematic structure to increase the number of problems which can be solved.

For example, we may erect the perpendicular at point C on line AB as follows (Fig. 2):

Draw $C(r)$ (the circle with center at C and with the fixed radius r) cutting AB in D. $D(r)$ cuts $C(r)$ in E, and $E(r)$ cuts $C(r)$ in F. $E(r)$ and $F(r)$ meet in G. Then CG is the desired perpendicular.

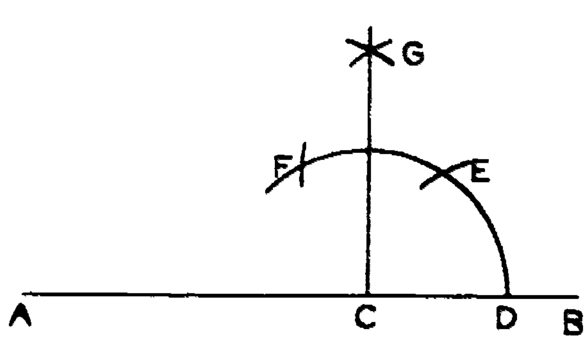


FIGURE 2

(An alternate method, using the properties of the 30°–60° right triangle, is to cut line ED with $E(r)$, thus giving G without the use of point F.)

The drawing of the line through point A which is parallel to given line BC can be performed by the "rhombus method" (Fig. 3):

$A(r)$ cuts BC in D. $D(r)$ cuts BC in E. $E(r)$ and $A(r)$ meet in F. Then AF is parallel to BC.

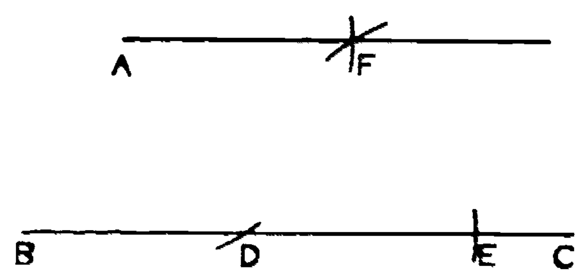


FIGURE 3

If the student attempts to drop a perpendicular from a given point, A, to a given line, BC, he will find that he has difficulty if A is at a distance greater than r from BC. However, by combining the two previous constructions (first draw any convenient perpendicular to the given line, and then draw a parallel to this line through point A), the difficulty is found to be only temporary.

At least four basic questions now present themselves: (1) What are the construction problems which can be solved using only the fixed-compass and straightedge? (2) What criteria can be set forth which will clearly define what is possible and what is impossible under such restrictions? (3) Why should such a problem be considered—either historically or as a present-day problem? and (4) Who are some of the persons who have found this problem to be of interest? The remainder of this paper is devoted to at least a partial answer to these questions. Rather obviously, these answers are interrelated, and no attempt will be made to consider them individually.

THE FIXED-COMPASS IN EARLY MATHEMATICAL HISTORY

Tradition has ascribed to Plato the responsibility of emphasizing the straightedge (of indefinite length) and the compass (of indefinite opening) as the basic tools for carrying out geometrical constructions. Euclid never used the word "compass" in his *Elements;* his first three postulates, however, appear to give emphasis to the idea that straightedge and compass were the only tools of pure geometry. The third postulate, "to describe a circle with any center and distance," and the manner in which it was used, result in a limitation which is usually expressed by saying that Euclid used a "collapsible compass." In effect, this compass closed as soon as one of its points was removed from the paper. Of course, Euclid immediately established proposition I-2: "to place at a given point (as an extremity) a straight line equal to a given straight line," and proposition I-3: "given two unequal straight lines, to cut off from the greater a straight line equal to the less." In effect, then, the devices of "transferring segments" or of marking off equal segments on a line by means of a single fixed opening of the compass were available, although Euclid did not give these duties to the compasses as such.

Present-day students are surprised at the way Euclid handles the basic constructions, but these are worthy of study to note the systematic and logical procedures which Euclid used. In the *Commentary on Euclid's Elements* written by Proclus about 400 A.D., there is evidence that other writers soon after Euclid used the compass as dividers for transferring distances. For example, Proclus gives a construction attributed to Apollonius (*ca.* 260–170 B.C.) for the method of drawing an angle equal to a given angle which is customarily used today. Proclus objected to such a construction, although it must be noted that he did so because of the demonstration (proof) of the construction involved, rather than because he objected to the actual method.

It seems significant to note this attitude of Proclus — it indicates much greater interest in the logical approach to geometry than in the practical approach. This attitude — avoiding the practical — probably accounts for the fact that we find in Greek geometry no real awareness of the fixed compass as a special device in performing constructions. If some of the basic constructions were obtained without changing the opening, this appears to have been done without any conscious placing of this restriction on the construction.[3]

[3]Several standard histories of mathematics have stated that Pappus (250–300 A.D.) made mention of such constructions with a single opening of the compass. That this passage in Pappus has been misinterpreted has been shown by W. M. Kutta, "Zur Geschichte der Geometrie mit Constanter Zirkelöffnung," *Nova Acta,* 71 (1898), pp. 72–74.

Similarly, credit has been given to Heron (first century A.D.) for a construction which was transmitted through a commentary on Euclid by the Arab, an-Nairizi (died 922). Heron's method can be adapted to a fixed compass construction, but his actual construction was not confined to a single opening

The constructions of Abû'l-Wefâ

It is in the work of an Arab of the tenth century that we find what seems to be the first recorded attempt to consider the problem of fixed-compass constructions systematically. In a work on *Geometrical Constructions*, which is ascribed to Abû'l-Wefâ (940–998), we find the explicit condition made in the statement of certain problems that the construction in each case is to be performed with a single opening of the compass.

A considerable number of mathematical works have been ascribed to Abû'l-Wefâ. He wrote commentaries on al-Khowârizmî, Diophantus, and Hipparchus; he began his own commentary on Euclid's *Elements* but apparently did not finish it. He wrote treatises on arithmetic and computation for practical use, computed tables of sines and tangents, and wrote extensively on astronomy.

Fixed-compass constructions are expressly called for in the statements of five different problems given by Abû'l-Wefâ. These are: (1) the construction of a regular pentagon on a given line segment as side, using only one opening of the compass equal to the given line segment; (2) the same, for a regular octagon; (3) the same, for a regular decagon; (4) inscribe a square in a given circle, using only one opening equal to the radius of the circle; (5) the same, inscribing a regular pentagon in a given circle.

In certain of these problems it is necessary to use auxiliary constructions, such as the erecting of perpendiculars and the bisecting of arcs, angles, and segments. In the earlier part of his work Abû'l-Wefâ actually included solutions for these problems which require only a single opening of the compass. The fixed-compass restriction is not stipulated in the statement of these problems, however.

The previously given method for erecting a perpendicular at a point on a given line (alternate method) was given by Abû'l-Wefâ. Following are some of the other constructions given by him (expressed in modern notation, but following the same steps).

To divide a line segment into any number of equal parts (e.g., 3) (Fig. 4): Erect a perpendicular at each end of the given segment AB in opposite directions. The opening of the compass is marked off twice from A and B, giving C and D, and E and F. CF and DE cut AB in points M and N, dividing AB into three equal parts.

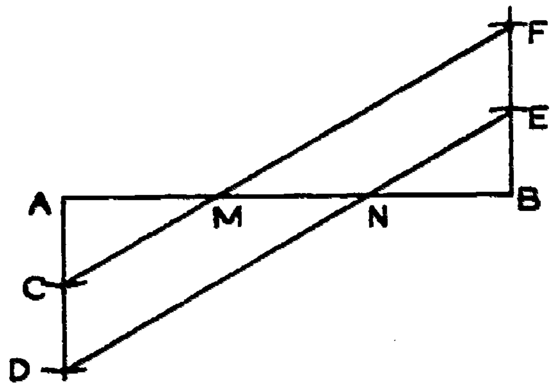


FIGURE 4

To inscribe a square in a given circle, using a single opening equal to the radius of the circle (Fig. 5): Given a circle with diameter AG and center S. A(r) gives Z, and $G(r)$ gives T. AT and ZG meet in M. Join MS, cutting the circle in B and D. Then $ABGD$ is the desired square.

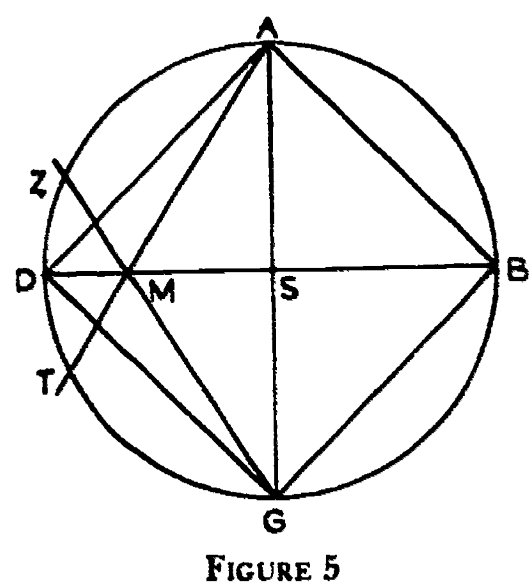


FIGURE 5

To construct a regular pentagon on given side AB, using only a single opening of the compass equal to the given side (Fig. 6): On given side AB, draw a perpendicular at B. On this perpendicular mark off $B(r)$, giving C. Find D, the mid-point of AB. Join CD. $D(r)$ gives S on CD. Find K, the mid-point of DS. At K erect the perpendicular CD, cutting AB extended in E. $A(r)$ and $E(r)$ meet in M. Join BM and extend beyond M.

$M(r)$ cuts this in Z. Triangle ABZ is the "triangle of the pentagon." $Z(r)$ and $B(r)$ meet in H, and $Z(r)$ and $A(r)$ meet in T, so that $ABHZT$ is the desired pentagon.

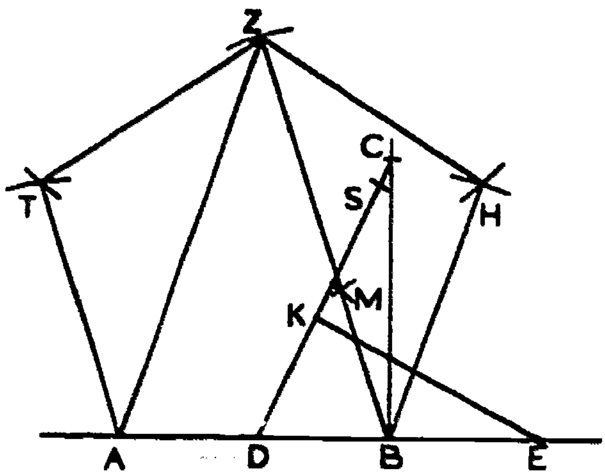


FIGURE 6

Why did Abû'l-Wefâ propose and solve these problems with a single opening of the compass? He gives no hint of the answer in his work. One conjecture has been that the compasses of that day were difficult to adjust. This seems difficult to defend when one recalls that the Arabs of this period were quite skilled in the construction of astronomical instruments and also had devices for drawing conic sections. Furthermore, there is evidence in the manuscript itself that it was not too difficult to change the opening of the compass. In an "artisan method" for drawing a parallel to a given line through a given point A, the center was placed at point A and the compass opening found which is the perpendicular distance from A to BC. Then any point on BC was chosen as center (see Figure 7) and an arc drawn. A line was drawn through A tangent to this arc to give the desired parallel—all of these steps being performed "by inspection."

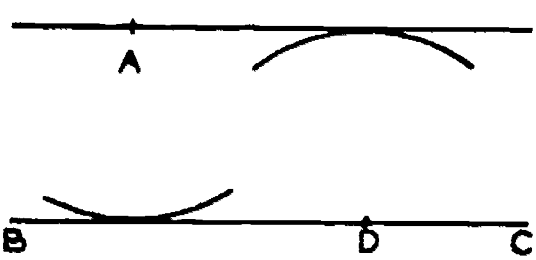


FIGURE 7

Another explanation might be that Abû'l-Wefâ was endeavoring to present a regular "theory" or systematic set of fixed-compass constructions. We will see that this endeavor did move later workers to consider the problem, but it seems quite clear that this was not the primary intent of Abû'l-Wefâ's work. The auxiliary constructions do not have the restriction stated explicitly; the fixed-compass constructions are scattered throughout the nonrestricted constructions (which greatly outnumber the former); and there are various non-fixed-compass constructions given by Abû'l-Wefâ which could easily have been converted to fixed-compass constructions if he had been committed to that purpose. Most important of all is the fact that the fixed opening in most cases is specified as being equal to some previously given length, such as the given side or the radius of the circle.

The most plausible explanation seems indicated when one notes that the fixed-compass constructions primarily are concerned with the drawing of regular polygons, both on a given side and inscribed in given circles. This, of course, is closely related to the similar problem of dividing a circle into any number of equal parts. This problem is an ancient one, involved in problems like that of determining the equal order of spokes in a wagon wheel, and in decorative and ornamental art work. Fixed-compass constructions were first developed to answer a practical need for regular polygon constructions in art, architecture, and the construction of scientific instruments. Fixed-compass constructions were more efficient, not because it was so difficult to adjust the compass, but simply because additional adjustments which would be necessary might prove more time-consuming and might possibly cause some inaccuracies.

It thus appears that interest in the fixed-compass geometry began, as in so many other topics in mathematics, in the attempt to find a practical solution to a common problem. It is striking to find the first explicit presentation of such restricted constructions as systematic and elaborate as it was. The ingenuity displayed in the construction of the pentagon is seldom surpassed in all of the later development of the geometry of the fixed-compass.

Continued interest in the practical problem

The next chronological references to fixed-compass constructions are to be found at the end of the fifteenth and the beginning of the sixteenth centuries. In three different works, written within a period of forty years, constructions are given in which the restriction that just a single opening is to be used is definitely stated. Here again the use of the fixed-compass as a practical device is rather clearly indicated.

Geometria Deutsch was a small printed work of just six pages without mention of author or of the time or place of publication. It is described as the "first printed outline on geometry in the German language"[4] and was probably printed before 1487. Nine geometrical constructions were given — without proofs — and in one of these the restriction *"mit unverrücktem Zirkel"* is made. This is the problem of constructing a regular pentagon on a given side with the fixed-compass opening equal to the given side. Unlike Abû'l-Wefâ's construction, however, this is only an approximation; on the other hand, it is much more easily executed.

Given segment AB, with $AB = r$ (Fig. 8). Draw $A(r)$ and $B(r)$, meeting in C, D. Join CD. $D(r)$ cuts CD in E, giving also F and G. EF cuts $B(r)$ in K. GE cuts $A(r)$ in H. $K(r)$ cuts CD in I. Then $ABKIH$ is the pentagon.

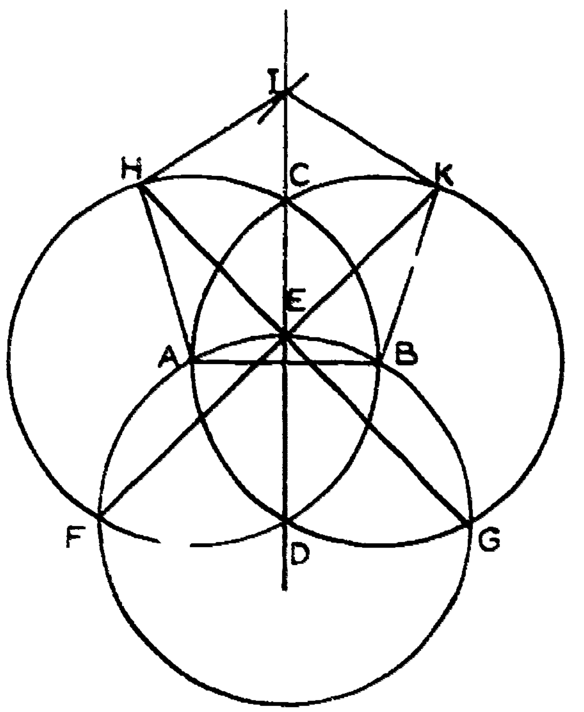


FIGURE 8

This same construction is given by Albrecht Dürer in his book of instructions on the art of measuring with compass and ruler, first published in 1525.[5] This work includes two constructions which are to be carried out with the *Zirkel unverrückt* — that given above and the inscription of a regular hexagon in a circle. The latter construction, of course, represents no special achievement. Dürer also includes several

[4] Siegmund Gunther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter bis zum Jahre 1525* (Berlin: A. Hofmann & Comp., 1887), p. 347.

[5] Albrecht Durer, *Underweysung der Messung mit dem Zirckel und Richtscheyt* (Nürnberg, 1525)

constructions involving just one opening of the compass, which may be arbitrary. One of these, which also appeared in *Geometria Deutsch*, is for constructing a right angle:

Draw any two lines meeting in E (Fig. 9). Place one point of the compass at E and swing an arc, cutting the lines at A, B, and C. Then AB is perpendicular to BC.

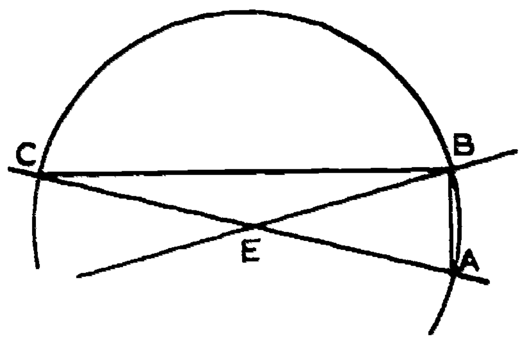


FIGURE 9

It will be noted that the construction simply gives a right angle and in no way indicates that the method could easily be adapted to the problem of constructing a perpendicular to a given line from a given point on the line. This "semicircle" method of constructing such a perpendicular is one of the most repeated fixed-compass constructions given by later writers. We jump ahead almost two hundred years to give the vivid description of an Englishman, William Leybourn:

"Set one point of your fork in the end B, and keeping it there, pitch the other end down upon the Paper at all adventures in C, and upon C turn the fork about till the other point of it touches the given line AB in D."[6] Then CD extended cuts the same arc again in E, and EB is the desired perpendicular (Fig. 10).

Of greater interest are the contributions of another great artist of this period, Leonardo da Vinci (1452–1519). In the so-called *Notebooks* of Leonardo are found recorded comments and sketches on statics and dynamics, anatomy, light and shade, architecture, perspective, and other topics — in varied order and in various stages of completion, usually in the left-handed manner of writing in "mirror-image." These manuscripts also contain geometrical constructions, and in particular we note that some form of the phrase *con una apritura di sesto* is found at least ten times scattered through these pages.

[6] William Leybourn, *Pleasure with Profit* (London, 1694), Tract II, pp. 16–27. The title page of this work was reproduced in William L. Schaaf, "Memorabilia Mathematica," THE MATHEMATICS TEACHER, XLVIII (February, 1955), 167.

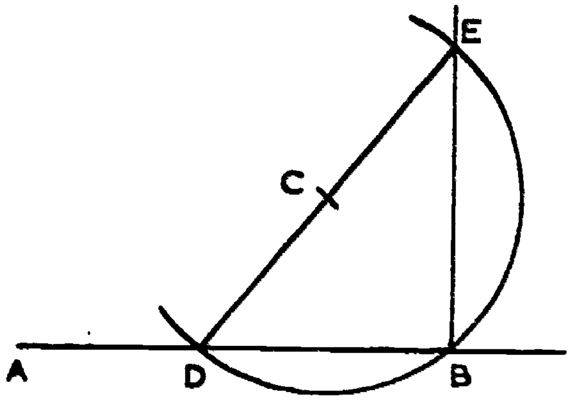


FIGURE 10

The diverse nature of Leonardo's fixed-compass constructions, together with the fact that many are incomplete and others are inaccurate, makes it difficult to systematize his attainments briefly.

It is evident that Leonardo used the above phrase with two different intents: first, the fixed-compass was to be used as dividers for laying off equal units of length along a straight line or the circumference of a circle, the proper opening being sometimes found by the "trial" method of adjusting, second, once given a particular opening, this was to be maintained throughout the entire construction, or at least until some sought-for distance was found, at which time a second fixed setting—not arbitrary—would be used as before in completing the construction.

A representative construction of Leonardo is that for dividing a circle into 3, 5, 6, and 30 equal parts.

Given a circle (Fig. 11) with A, B, C, D points of the inscribed hexagon. $D(r)$ cuts AD in N. BN cuts the circle again in M. Then, $AD = \frac{1}{3}$, $AM = \frac{1}{5}$, $AC = \frac{1}{6}$, $CM = 1/30$ (of the circumference).[7]

Here the sides of the pentagon and hence of the 30-gon are only approximate. In other constructions Leonardo proposed to divide circles into 3, 7, 8, 9, 18, and 24 equal parts. Actually, only the lengths of the sides of the required n-gons were found, and the implication here is that the compass would have to be reset to the proper opening and the points then stepped off around the circumference. Some of the above lengths are only poor approximations—so much so that Leonardo sometimes wrote "*falso*" next to the construction.

[7] Leonardo da Vinci, *Les Manuscrits de Leonardo da Vinci*, (6 vols.; Paris: Charles Ravaisson Mollien, 1881–1891), B-27 v.

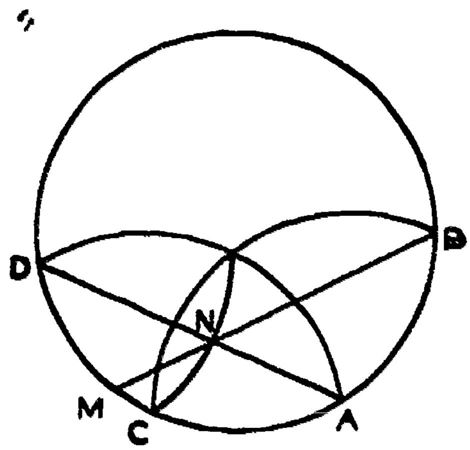


FIGURE 11

A detailed study of Leonardo's constructions leads one to the conclusion that Leonardo considered the fixed-compass as a convenient tool for the artist and architect and not as a device of theoretical mathematical interest. Realizing that there are natural limitations to the accuracy of all such constructions anyway, Leonardo seems only to have been concerned as to whether a construction was accurate enough for his purpose; he was not interested in its formal mathematical proof. He used freely, without acknowledgment, the common knowledge available to the artist and engineer of that time, and he continued to experiment for easier and simpler ways of obtaining practical results. This meant that he accepted the fixed-compass as one of his drawing instruments and used it among others in looking for new methods. There seems little reason to believe that Leonardo himself passed on these constructions to his contemporaries or to his successors.

The similarities in these three works lead one to speculate over possible reasons for this. The mathematical historian, Moritz Cantor, considered the question of whether Dürer had access to *Geometria Deutsch;* he concluded that Dürer did not, because of the absence of some of the earlier constructions in Dürer's work. Rather, Cantor felt that such constructions as these were used by the architects and builders of that time, perhaps passed on secretly by them from one generation to another.[8]

The significant conclusion that can be drawn from the presence of these constructions in these three works is that they reflect a common body of practical geometrical knowledge known and used by the artist and artisan of that day. The fixed-compass was one of their tools—which was used frequently but certainly not exclusively.

[8] M. Cantor, *Vorlesungen über Geschichte der Mathematik* (2d ed.; Leipzig, 1913), II, pp. 461, 465.

THE SOLUTIONS OF THE 16TH CENTURY ITALIANS

We have noted above how interest in the practical aspects of geometry had led to fixed-compass solutions for regular polygon constructions, with the fixed opening being equal to the given side or the given radius. In the middle of the sixteen century, a complete change occurred in the geometry of the fixed-compass—a change both in motivation and in the nature of the problem considered. Within a period of ten years there appeared three different sets of fixed-compass solutions for all of the construction problems given in Euclid's *Elements*. Moreover, the opening of the compass was completely arbitrary—actually, it was to be an opening "proposed by the adversary."

The controversy over the discovery of the general solution of the cubic equation has long been one of the interesting episodes in sixteenth century mathematics, although there have been difficulties in determining the exact details.[9] No attempt will be made here to reconstruct the details of this bitter controversy between Nicolo Tartaglia and Hieronimo Cardano after Cardano published in the *Ars Magna* the solution of the cubic supposedly received from Tartaglia under oath of secrecy. In 1547 Ludovico Ferrari issued his first *Cartello* in defense of Cardano (his teacher and benefactor) and gave challenge to Tartaglia for a mathematical duel in which each would propose thirty-one problems to be solved by the opponent.[10] In the *Seconda Riposta* Tartaglia proposed his thirty-one questions, the first seventeen of which dealt with problems to be solved by means of the fixed-compass. Included were such problems as these: to draw a tangent to a given circle from an outside point; to describe a rectilineal figure which is similar to a given figure and equal (in area) to another; to construct a triangle with angles in the ratio of 2:3:10; to find the tangent to an ellipse which makes an angle with the major axis equal to a given angle.

Ferrari submitted his answers some months later in the *Quinto Cartello*. Actually, instead of answering the specific questions proposed by Tartaglia, Ferrari presented a complete treatment of Euclid's first six books, with the change in the third postulate that the opening of the compass was to remain fixed at an arbitrary opening throughout. The propositions were necessarily presented in different order from that given by Euclid, but Ferrari was careful to use at any given time only those

[9] Oystein Ore, *Cardano* (Princeton University Press, 1953), presents the most complete account of the Tartaglia-Cardano controversy, however, it makes no reference whatsoever to the fixed compass problem.

[10] The letters exchanged were reprinted in facsimile in a limited edition. L. Ferrari and N. Tartaglia, *Cartelli e Risposte* (Milano: F. Giordani, 1876).

constructions or theorems which he had previously established in his own particular sequence. Frequently a number of propositions were grouped together with the note that these could be carried out as in Euclid.

In Ferrari's introduction to his solutions he comments that he did not know who first proposed this principle of working without changing the opening of the compass, but that for the last fifty years many had worked on this problem, including particularly Scipione dal Ferro. The "many" are not identified, but this indicates a definite interest in this problem extending over a period of time. No further information on the role of Ferro has been found. It is known that in 1543 Ferrari and Cardano examined the papers of Ferro (who had died in 1526) in connection with the solution of the cubic equation. It has been conjectured that at that time they found some reference to the fixed-compass problem in Ferro's papers.

Another interesting question is that of what induced Tartaglia to present such problems. One can assume that Tartaglia was already in possession of solutions for the questions he proposed to Ferrari when he submitted them. Whether he at that time had actually thought of doing "all of Euclid" is another matter. In 1556 Tartaglia published his *General Trattato di Numeri et Misure;* included in this was his own set of solutions for all construction problems in Euclid and his description of how his own interest in the problem originated. He refers to a remark of Aristotle that in any given art one should look beyond the usual meaning for something to admire, for something intelligent and different from the others. Hence one day he turned to proposition VI-25 of Euclid (the second of those described above which were given to Ferrari), to see if it could be resolved with any opening of the compass proposed by an adversary. He soon found that this was possible — in fact that it was possible to solve all of Euclid's problems which are worked in a plane, with the exception of those which involve the drawing of certain specified circles with radii unequal to the fixed opening.

It must be observed, however, that this statement appeared after two other sets of fixed-compass solutions of Euclid's propositions had been published. In 1550 Cardano published the twenty-one books of the *Subtilitate* and in Book XV presented a brief condensation of his and Ferrari's work. Only twelve constructions are included, although they are the most significant ones. Tartaglia's name was not mentioned, which may be assumed to be not merely an oversight!

In 1553 Giovanni Battista Benedetti (the Latinized form is Ioannes Baptista de Benedictis) published in a booklet of over 130 pages a complete set of fixed compass solutions for the problems of Euclid. There is no reference to the beginnings of his interest in the problem. Benedetti had been a student under Tartaglia and states in his introduction that he had studied the first four books of Euclid with Tartaglia — the rest he had studied privately. Benedetti published his work at the age of 23, so that his contact with Tartaglia must have come soon after the time of the *Cartelli-Risposte*. There is no mention of the controversy or of the published accounts of Ferrari or Cardano.

A reasonable conclusion on the basis of the presently known facts is that Tartaglia and Ferrari-Cardano independently worked out their basic solutions; that Benedetti worked out the details of his sequence after having obtained some ideas on the nature of the problem from Tartaglia and that some of the details in the Tartaglia sequence may have been influenced by either or possibly both of the earlier published sets given by Ferrari-Cardano and Benedetti. This is probably another example in the history of mathematics of the fact that after certain preliminary ideas have become common knowledge, the final steps may be taken independently by several individuals.

The "All of Euclid" criterion

The inquisitive student by this time will have raised the question of whether the fixed-compass and straightedge are "equivalent" to the ordinary compass and straightedge. Is it possible in this limited manner to perform all constructions that can be performed by the traditional means? The accompanying problem, of course, is how such an equivalence could actually be established. There is no direct evidence that the Italians believed that by establishing "all of Euclid" they had thereby proved the fixed-compass and straightedge to be equivalent to the ordinary compass and straightedge. (In fact, it is a matter of speculation whether the questions of such equivalence were really of any concern to them.) Certainly, however, they recognized that the fixed-compass was a possible, if awkward, tool for "performing all of Euclid," and probably that is as much as could be expected of them at this stage of development in mathematics.

The interest that had been climaxed by the appearance of these three sets of solutions, all within a period of ten years, died down abruptly. The "practical" uses of the fixed-compass constructions were not extended by the solutions of Euclid (which would often refer back in chain-line succession to constructions previously described). And the theoretical aspects of fixed-compass constructions could have no new points of emphasis until the field of geometry could be extended. An important chapter in the development of the geometry of the fixed-compass was clearly ended.

Soon another chapter began, but, strange to say, historically and mathematically it was almost a complete rewrite of the previous chapter. The previous works came too early to be widely disseminated, except for the *Subtilitate;* here the constructions were so condensed and hidden as to attract little attention. On the other hand, the increased use of the printing press and the greater concern for at least the elements of an education made geometry a matter of interest to many more individuals than before. It was natural that to many such persons the fixed-compass constructions (on a more elementary level) would have an appeal because of their practical, novel, and puzzle-like aspects. As an added incentive, enough of the past achievements of the sixteenth-century Italians were passed along, although primarily as heresay, to encourage work on some aspects of the problem. Ultimately, these endeavors were again to be climaxed by a set of fixed-compass solutions of "all of Euclid," apparently arrived at independently.

THE PERIOD OF REDISCOVERY

It is possible to point to references to the fixed-compass in the work of at least thirteen writers in this "period of rediscovery." (We would date this period from 1560 to 1700 — fixed-compass work is almost completely absent from any writings of the eighteenth century.) We shall indicate the work of several representative persons with their respective approaches.

The seventeenth century brought forth many printed geometries in various languages. Some of these were commentaries or simplifications of Euclid's *Elements;* others emphasized the practical aspects of geometry. In 1613 Pietro Antonio Cataldi published in Italian the first six books of Euclid "reduced to practice." Cataldi thus included sections headed *In Practica* for each of Euclid's propositions. In discussing Euclid's first proposition, Cataldi states: "We add, with pleasure, certain operations which have been changed from the ordinary, in order that the student may receive delight in establishing them, as well as the desire to study and follow with attention and diligence; as for example in the first proposition or problem, to carry out the demonstration also with one given aperture of the compass, which is smaller or larger than the given line: to erect an equilateral triangle on the given line." For this problem Cataldi simply gives two drawings (see Figure 12) and says, "let the figures speak for themselves."

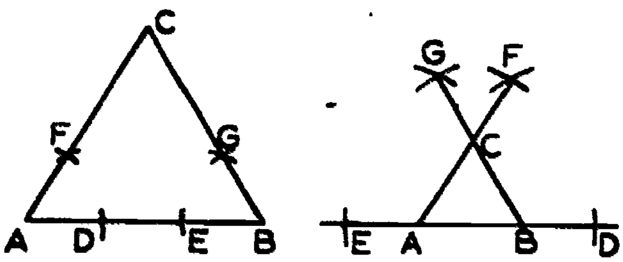


FIGURE 12

After such an introduction one might expect that a large number of fixed-compass constructions would be given. Actually, only four more elementary constructions are included; Cataldi gives no evidence of being interested in the ultimate possibilities of the fixed-compass or of having studied the fixed-compass constructions of the sixteenth-century Italians.

Mario Bettini published a three-volume work in 1648, entitled *Aerarium Philosophiae Mathematicae*, which considered at great length the propositions of Euclid's first book. Bettini's work includes references to several of the constructions of Cataldi and also to the Latin commentary on Euclid given by the German Christoph Clavius, published in 1574. It is interesting to note that Bettini devoted a special paragraph, well along in his first volume, to the fact that he had given six constructions which could be carried out with *una circini diductione*. These six constructions are all elementary, and again it seems evident that Bettini did not know of the work of Ferrari and his successors. Bettini included the "rhombus" method for finding the parallel to a given line through a given point; this method he attributes to Clavius, who appears to be the first person who recorded it.

In 1616 a somewhat insignificant German mathematician, Daniel Schwenter, began the publishing of individual portions of a *Geometrie, Tractatus I, II*, etc. These were later combined to form the *Geometriae practicae et aucti* which went through several editions. Schwenter knew of several isolated fixed-compass constructions (such as that of the pentagon usually ascribed to Dürer); he knew that Cardano and Tartaglia had worked on this problem, although it is evident that he was not aware of the full significance of their accomplishments.

Schwenter's work represents an interesting mixture of the approaches we have already noted. We find him presenting "puzzle problems" based on the fixed-compass; he gives what he indicates to be a practical use of the fixed-compass device; to a slight degree he considers the theoretical problem involved. In the latter case he divided a circle into any number of equal parts (from 2 to 10) *mit unverrüctem Circkel*, with the opening equal to the radius of the given circle. Schwenter recognized that the sides of the 7-gon and 9-gon were only approximate.

In the *Geometriae* Schwenter relates that at one time his tutor, Johann Praetorius, proposed the problem whether it was possible, *mit unverrücktem Circkel*, at one center and with one drawing, to draw an "oblong circle." Schwenter gives the following solution: place the paper on a cylinder or column, place the compass on the paper, and then draw a "circle," keeping the paper always in contact with the cylinder. Upon removing the paper from the cylinder, a "neat" oval is obtained.[11]

Schwenter also includes the following puzzle problem, which we will give in the form found in William Oughtred's *Mathematical R · eations*: "With one and the same compasses, and at one and the same ex ent, or opening, how to describe many Circles concentricall, that is, greater or lesser one than another? In the judgment of some it is thought impossible: who consider not the industrie of an ingenious Geometrician, who makes it possible, and that most facill, sundry wayes; for in the first place if you make a Circle upon a fine plaine, and upon the Center of that Circle, a small pegge of wood be placed, to be raised up and put down at pleasure by the help of a small hole made in the Center, then with the same opening of the Compasses, you may describe Circles."[12]

The account given by Schwenter seems to indicate that he felt this could be put to some practical use: he speaks of using a peg on the work table of a joiner (cabinet maker); by experimentation, the peg could be lowered or raised by hammering it in or out to make a circle of proper radius.

Several later writers refer to the fixed-compass constructions given by Schwenter. He seems to have thought of the fixed-compass primarly as a puzzle device — as an instrument which gives rise to interesting speculations and exercises. He was able to acquire this interest from those before him and to pass it on to those after him, and thus he played an active role in the development of the fixed-compass geometry.

[11] This same problem appears in various editions of the French *Récréations mathématiques* which is ascribed to Jean Leurechon, although this title sometimes appeared under the pseudonym of H. van Etten or without an author given. The earliest edition seems to have appeared in 1624. Whether this problem goes back to the same common source, or whether it was independently arrived at by Praetorius and Leurechon, is not known. It is repeated in many later books on mathematical recreations.

[12] William Oughtred, *Mathematical Recreations* (London, 1653), pp. 49–50. This was an English translation of Leurechon's work. Schwenter gives this problem in another work, *Deliciae Physico-Mathematicae* (Nurnberg, 1636), p. 131, and indicated it was not his own.

Fixed-compass constructions in "Euclidis Curiosi"

We turn next to one more important work in this period of rediscovery, the Dutch *Compendium Euclidis Curiosi*, published anonymously in Amsterdam in 1674. The work was translated into English and printed by Joseph Moxon in England in 1677. The subtitle in English reads: "Geometrical Operations, how with one given opening of the compass and a ruler all of the works of Euclid are resolved." In the preface the unnamed author states that he had read that one John Baptista had performed all of Euclid's propositions with one single opening of the compass. He had never found this work or any additional reference to it, although he had found this problem considered in the works of Bettini and Schwenter. Although he at first thought the matter impossible (especially when he considered such a problem as constructing a triangle given the three sides), he had studied it at length and was now able to present a set of solutions for all construction problems in Euclid.

A detailed study of the *Euclidis Curiosi* requires more space than can be devoted to it in this paper. Most important, however, is the fact that authorship of this work can now definitely be ascribed to Georg Mohr, the Danish mathematician whose name is familiar to those readers who are acquainted with the "geometry of the compasses alone."[13] For it was this same Georg Mohr who in 1672 had published the *Euclides Danicus*, the work which first gave proof that all constructions of Euclid could be performed with the (movable) compasses alone, without the use of a straightedge. (Such constructions are commonly referred to as "Mascheroni constructions" since the Italian, Lorenzo Mascheroni, independently established the same results in 1792.) Mohr's work on the "geometry of the compass" was not known until a copy of the *Danicus* was rediscovered in 1928.[14]

Mohr presents twenty-nine basic constructions which include all of the main construction problems of Euclid; proofs are omitted. The last problem concerns the drawing of plans for laying out a regular fortification, using fixed-compass and straightedge.

We include Mohr's construction for adding and subtracting segments:

[13] The copy of Moxon's English translation of the *Curiosi* in the Library of the University of Michigan was first pointed out to me by Professor Phillip S. Jones. Later Professor Jones saw the copy of the anonymous Dutch edition in the Plimpton Collection in the Library of Columbia University and aided in obtaining a microfilm of this work. For details on why the authorship can be ascribed to Mohr, see Arthur E. Hallerberg, "The Development of the Geometry of the Fixed-Compass with Especial Attention to the Contributions of Georg Mohr" (Ed.D. dissertation, University of Michigan, 1957, L. C. Card No. Mic. 58–1409).

[14] For example, see Julius H. Hlavaty, "Mascheroni Constructions," The Mathematics Teacher, L (November, 1957), 482–487, and N. A. Court, "Mascheroni Constructions," The Mathematics Teacher, LI (May, 1958), 370–372.

To join CD to AB, at B in line with AB (Fig. 13): Through B draw BK parallel to CD, and through D a parallel to BC, giving E. $B(r)$ gives H and F. Draw EP parallel to HF. Then P is the desired point.

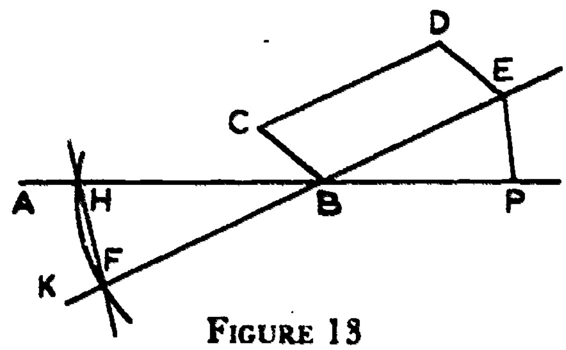


FIGURE 13

The necessary auxiliary construction had previously been given by Mohr. He also gives some special cases of the above problem.

Mohr seems to indicate that he believed the fixed-compass with straightedge was equivalent to ordinary compass and straightedge: in his preface he states that he could have added more operations such as for drawing sun dials, but "considering that all flat or plain operations may be reduced from these [operations given in this work], these shall suffice."

The "John Baptista" given in the preface was G. B. Benedetti; it is not at all strange that Mohr never located his work. A detailed analysis of Mohr's constructions as well as those of the Italians clearly indicates that Mohr obtained his results without access to their work.

Again, it is interesting to note the repetition of mathematical history in the almost complete disappearance of this set of geometrical solutions to the fixed-compass problem. Very little mention of either the Dutch original of *Euclidis Curiosi* or of the English translation by Moxon is to be found in references to the geometry of the fixed-compass.

WILLIAM LEYBOURN'S "PLEASURE WITH PROFIT"

Of lesser significance, but still of interest, are the contributions of an Englishman in 1694. Among the "mathematical practitioners" of England in the seventeenth century was William Leybourn. He seems to have found the same fascination in the novel and practical aspects of the fixed-compass which had aroused the interest of amateur mathematicians earlier in the same century. In his work, *Pleasure with Profit*, a section is devoted to "Geometrical Recreations." Chapter II of this is subtitled: "Shewing How (Without Compasses) having only a common Meat-Fork (or such like Instrument, which will neither open wider, nor shut closer), and a Plain Ruler, to perform many pleasant and delightful Geometrical Operations."

Leybourn gave twenty constructions for some of the more elementary problems. There is a decidedly practical flavor to some of these, and others seem to be included as novelties or curiosities. Leybourn does not attempt "all of Euclid" and there is no indication that he had examined any of such sets of solutions.

THE EQUIVALENCE OF THE FIXED-COMPASS AND THE ORDINARY COMPASS

While the fact that "all of Euclid" could be performed with fixed-compass and straightedge had been repeatedly established, it is obvious that a more refined set of criteria must be specified to actually establish the equivalence of the fixed-compass and the ordinary compass. Such criteria were presented in connection with the important Poncelet-Steiner theorem which is indirectly associated with the fixed-compass problem. This theorem may be stated in this form: If a single circle and its center are once drawn in a plane, every construction possible with ruler and compass can be carried out with the ruler alone. This theorem is ascribed to two persons, Victor Poncelet, who first stated the theorem and indicated a method of proof in 1822, and Jacob Steiner, who gave a systematic and complete presentation of the problem in 1833.[15]

Since an arbitrary fixed-compass can be used just once to draw the necessary "Poncelet-Steiner circle," it follows that the fixed-compass and straightedge are equivalent to the ordinary compass and ruler. The proof of the theorem depends upon somewhat more advanced geometrical concepts, but Poncelet gave the necessary criteria. Essentially, he pointed out that the ordinary compass and ruler can be used to find: (1) the intersection of two straight lines; (2) the intersections of two circles; and (3) the intersections of a line and a circle. The checking of such "intersection" criteria can be used to establish the equivalence of various geometrical tools with the ordinary compass and straightedge. There is little reason to believe, however, that the motivation for the Poncelet-Steiner theorem came from the fixed compass problem as such. Instead, we find an example of the transfer of knowledge from one phase of mathematics to another seemingly unrelated topic.

It is of course possible to apply the "intersection criteria" directly to fixed-compass constructions without reference to the Poncelet-Steiner theorem.[16]

[15] J. Victor Poncelet, *Traité des propriétés projectives des figures* (Paris, 1822); Jacob Steiner, *Die geometrischen Konstructionen* . . . (Berlin, 1833); M. E. Stark (trans.), R. C. Archibald (ed.), Jacob Steiner's *Geometrical Constructions* . . . (New York: Scripta Mathematica, 1950).

[16] See, for example, M. F. Woepcke, "Recherches sur l'histoire des sciences mathématiques chez les orientaux," *Journal Asiatique*, Vol. 5, Series 5 (1885), pp. 218–256, 309–359; also, K. Yanagihara, "On Some Methods of Constructions in Elementary Geometry," *Tôhuku Mathematical Journal*, XVI (1919), 41–49. The article by Woepcke is the main source of information on Abû'l-Wefâ.

THE "ANALYTIC CRITERION" AND PROJECTIVE GEOMETRY

Within only a little more than the past century, the use of analytic means of combining algebra and geometry has supplied criteria for what is impossible with the ordinary compass and straightedge as well as for what is possible; the same limitations and possibilities were thereby automatically set for instruments which are equivalent to them. A discussion of this matter would lead us too far afield from our original purpose. Essentially, however, this consists in showing how ruler and compass can be used in performing the four fundamental operations and the extracting of the square root of certain positive numbers, providing that an appropriate co-ordinate system has been set up. Particular geometrical problems are then related to the roots of certain equations, and the constructibility of these roots can then be considered.

In this paper we have confined ourselves to the fixed-compass problem as it is related to elementary geometry. A detailed discussion of the Poncelet-Steiner theorem and its various modifications would lead us to many of the concepts studied in projective geometry. We indicate just one modification of this theorem.

In 1934 Mordoukhay-Boltovskoy proposed the question: if the pair of compasses broke before the entire Poncelet-Steiner circle had been completed, what would one do in this "catastrophe"?[17] Actually that question had been answered thirty years before, when Francesco Severi proved the theorem, "All problems solvable by ruler and compass can be resolved with ruler and a traced arc of a circle, with center given."[18] In relation to the geometry of the fixed-compass, this means that the fixed-compass need not be used more than once, and then only a portion of the circle need be drawn. It should be emphasized that the center of the arc or circle is necessary.

"TO THE INTERESTED READER"

The "joy of discovery" in mathematics is, of course, one of the satisfying characteristics of its study. For the present we therefore leave as a problem "for the interested reader" the drawing of fixed-compass solutions of other customary compass and ruler constructions. The problem of constructing a triangle, given its three sides, by means of the fixed-compass, has fascinated amateur mathematicians for over 500 years, and it may well do so for many more years to come.

[17] D. Mordoukhay-Boltovskoy, "Sur les constructions au moyen de la règle et d'un arc de cercle fixe dont le centre est connu." *Periodico di Matematiche* (Series 4), XIV (1934), 101.

[18] F. Severi, "Sui problemi determinati resolubili colla riga e col compasso." *Rendiconti del Circolo Matematico di Palermo*, XVIII (1904), 256-259.

It is by no means a new idea to consider constructions carried out with tools other than the customary straightedge and compass an appropriate topic in high school and college mathematics. The concept of "constructions with limited means" has been used to refer to a restriction on the geometrical tools used or to restrictions on the manner in which they are used. Thus attention has been given to the constructions which can be carried out with the compass alone, with the ruler alone, with a parallel ruler, with the right angle, or the like.

The point that does not seem to have been emphasized in the past is that the fixed-compass, of all limited means, is of particular interest and value. We therefore conclude with the following features which particularly distinguish the geometry of the fixed-compass: (1) the student already knows several constructions which use the principle of a single opening, and so the question of what can be accomplished in this way is a natural one; (2) it is based on elementary geometrical concepts — there is no need to develop new or special topics (such as inversion, for the geometry of the compass alone) before the student can work on the problem; (3) many of the basic constructions are within the attainment of even the average student; (4) the fixed-compass is the oldest of the limited means, and its history is by far the richest; (5) some of the most important motives which have led people to do mathematics are clearly discernible in its development: the motives of the practical, of the puzzle, of intellectual curiosity, and of abstract theory; (6) it is associated with persons who are of interest for other reasons: Abû'l-Wefâ, Leonardo, Dürer, Cardano, Tartaglia, and others; (7) the continuity of growth of mathematics is illustrated in the independent repetitions of certain approaches and results of different persons; (8) some of the desirable "appreciations" which may result from a study of mathematics are included here — the contrasts of simple and advanced mathematics, of utility and abstract theory, of approximation and exactness, of conjecture and demonstrative proof, of puzzles and practicality, of possibility and impossibility, of generalization and specialization; and (9) the fixed-compass leads into more advanced topics in projective geometry, indicating some of the interrelations in mathematics.

Certain Topics Related to Constructions With Straightedge and Compasses

Adrien L. Hess

Introduction

Closely related to the problem of geometric constructions are certain topics which serve to extend and enrich the usual conception of such constructions. The topics represent various facets of the problem which have been developed within the last one hundred sixty years. Of such topics, the three most closely related to geometric constructions are: Geometrography, Paper-folding and Match Stick Geometry.

Geometrography

In 1833 Steiner (17), an outstanding German mathematician, suggested that every construction in geometry should be studied so that the solution used would be the simplest, the most exact, and the surest. He also proposed that this study should include constructions in general, and constructions made under limitations as to instruments used and with obstructions existing in the plane. Nothing seems to have materialized from the outlining of this problem until, in 1884, Wiener solved several constructions for which he counted the number of circles and straight lines drawn (2).

Lemoine, who made the first systematic approach to the problem, presented his initial ideas to the leading French scientific society of his time in 1888. In less than fifteen years he wrote more than thirty notes and memoirs, which appeared in many mathematical and scientific journals, in which he amplified and extended his ideas on geometrography. Starting in 1888 with geometrography as applied to straighedge and compasses construction, he had extended his system by 1894 to include descriptive geometry and by 1902 to include geometry of three dimensions (9; 10). In 1902 Lemoine summed up his development of geometrography in his book *Géométrographie, ou Arts des Constructiones Géométriques* (11).

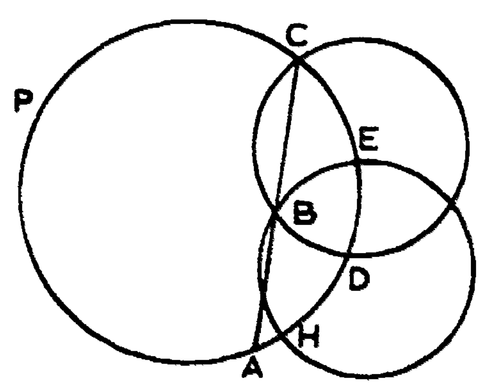
Although geometrography was developed mainly by Lemoine, its growth was aided by the contributions and comments of many writers in England, France and Germany. Other systems of geometrography were devised by Papperitz (13) in 1908 and by Grüttner (8) in 1909. Lemoine. Godeaux (7) and Adler (1) extended the system to include tools other than the straightedge and compasses. In 1929 Tuckey (18) devised a system of geometrography in which he considered only the settings of the straightedge or compasses and the number of straight lines or circles drawn. Some recent college geometry textbooks (4;16) include a brief discussion of geometrography.

Lemoine chose two operations for the straightedge and three operations for the compasses as the fundamental operation in his system of geometrography. The five operations and their symbols are:

1. To place the straightedge on a given point . . . . R_1
2. To draw a straight line with a straightedge . . . . R_2
3. To place one point of the compasses on a given point . . . C_1
4. To place one point of the compasses on any point of a line . C_2
5. To draw a circle . . . . C_3

If in a construction, these operations occur respectively a_1, a_2, b_1, b_2, b_3 times, the symbol for the construction is $a_1R_1 + a_2R_2 + b_1C_1 + b_2C_2 + b_3C_3$. The total number of operations is the sum $a_1 + a_2 + b_1 + b_2 + b_3$, which is called the coefficient of simplicity *(S)*. The sum $a_1 + b_1 + b_2$ is the total number of coincidences and is called the coefficient of exactitude *(E)*.

The system will be illustrated by Swales' Construction for finding the radius of a circle when the center O is not given.



With any point O on the given circle $O(P)$ and any convenient radius r, draw circle $D(r)$ to intersect $O(P)$ at H and E . . . . $C_2 + C_3$

With E as center draw circle $E(r)$ to intersect $O(P)$ at C and $D(r)$ at B . . . . $C_1 + C_3$

Draw the straight line BC to intersect $O(P)$ at A. AB is the radius of the circle $O(P)$ . . . . $2R_1 + R_2$

The symbol for the entire construction is $2R_1 + R_2 + C_1 + C_2 + 2C_3$. The coefficient of simplicity is $S = 2 + 1 + 1 + 1 + 2 = 7$. The coefficient of exactitude is $E = 1 + 1 + 2 + 4$. The construction with the smaller coefficient of simplicity is considered the simpler construction.

Paper Folding

Although, historically, the folding of materials and the making of knots are quite old, their application to geometry has been made in more recent times. It was about five hundred years ago that the great German, Albrecht Dürer, who was interested in geometry as well as art, first showed that the regular and semi-regular solids could be constructed out of paper by marking the boundaries of the polygons, all in one piece, and then folding the polygons along the connected edges (3). The first English translation of Euclid's *Elements*, printed in 1570, included a most interesting feature. In the eleventh Book of the translation, figures made of paper were pasted in such a way that they could be opened up to make actual models of space figures (14). Over a century later Urbano D'Aviso, a student of Cavalieri, published a work in Rome entitled *Trate de la Sphere*, in which geometric constructions were worked out by means of paper folding. The formation of a regular hexagon and a regular pentagon by means of knots, a type of paper folding, is attributed to him (6).

In 1893 two men of different nationalities and in widely separated countries wrote works on paper folding. Wiener, a teacher in a German polytechnic school, showed how to construct regular convex polyhedra by paper folding (2). Row, a mathematician of India, wrote a book in which he gave a more complete treatment of paper folding (14). This work was translated by Beman and Smith in 1901 and the book became readily available in this country.

In 1905 another book, entitled *First Book of Geometry*, appeared which used paper folding. The authors Grace C. Young and W. H. Young, feeling that Row's book was too advanced for children and too puerile for adults, wrote their book to meet the needs of children. In 1908 it was translated into German under the title *Der Kleine Geometer* (20). The book is designed to give instruction to young children in fundamental ideas of plane and solid geometry. No particular apparatus is needed for the constructions chosen and these constructions can be made and understood by children four and five years of age. Besides the usual fundamental constructions of geometry, other constructions are given in the book to develop understanding of the concept of inequality, regular polygons, parallel lines and planes, and the theorem of Pythagoras.

As shown by Yates (19) with properly chosen postulates, all constructions of plane geometry that can be carried out with a straightedge and compasses can be executed by paper folding.

Match Stick Geometry

Match stick geometry, devised by Dawson (5) in 1939, uses as its sole tool a finite supply of match sticks of equal length. For his geometry he chose four postulates:

1. A straight line may be laid to pass through a given point, or with one extremity on a given point.
2. A line may be laid to pass through two given points, or with one extremity at one given point and passing through a second point, but the two points may not be such as lie in a given line or laid line.
3. A line may be laid with one extremity at a given point and its other extremity on a given line.
4. Two lines may be laid simultaneously to form the sides of an isosceles triangle, two of their extremities coinciding and the other two being given points.

Two lemmas and an assumption complete the geometry. The lemmas are: A given line of a length less than, equal to, or greater than the length of a match stick can be bisected; a line can be laid through a given point and parallel to a given line. Since a circle cannot be drawn, it is assumed that a circle is determined when its center and a point on the circumference are given.

The construction of a half hexagon is characteristic of the operations of this geometry. The equilateral triangle ABC is constructed. On side BC the equilateral triangle BCD is constructed with D distinct from A. On side DB the equilateral triangle DBE is constructed to form the half hexagon $ACDE$. Thus AB is extended in a straight line so that $AE = 2AB$. This construction also gives a way of constructing a line parallel to a given line, for CD is parallel to AB.

Under the postulates and the assumptions stated above, it is possible to perform all constructions which are possible with a straightedge and compasses.

Bibliography

1. Adler, A. *Theorie der Geometrischen Konstruktionen*, Leipzig, 1906.
2. Archibald, R. C. "Geometrography and Other Methods of Measurements of Geometry," *The American Mathematical Monthly*, 27:323–326, July–Sept. 1920.
3. Cajori, F. *A History of Elementary Mathematics*, New York: The Macmillan Company, 1917.
4. Daus, P. H. *College Geometry*, New York: Prentice Hall Publishing Company, 1941.

5. Dawson, T. R. "'Match Stick' Geom. try," *The Mathematical Gazette*, 23:161–168, 1939.
6. Fourrey, E. *Constructions Geométriques*, Paris: Vuibert, 1924.
7. Godeaux, L. "Applications des Methods Géométrographie au Tracé Mechanique des Courbes Planes," *L'Enseignement Mathématique* 8:143–146, 370–373, 1906.
8. Grüttner, D. "Die Zerlegung geometrischer Zeichnungen in Konstructionen elements und ihr Anwendung bei der Lösung von Aufgaben," *Zeitschrift für Mathematischen und Naturalwissenschaftlichen Unterricht*, 39:256–261, 1908.
9. Lemoine, E. "La Géométrographie dans L'Espace ou Stereométrographie," *Mathesis*, 22:105–107, 1902.
10. .... "Principes de Géométrographie ou Art des Constructions géométriques," *Archiv der Mathematik und Physik*, 1(3):99–115, 323–341, 1901.
11. .... *Géométrographie ou Art des Constructions Géométriques*, Gauthiers-Villars, 1902.
12. McKay, J. S. "The Geometrography of Euclids' Problems," *Proceedings of the Edinburgh Mathematical Society*, 12:2–16, 1894.
13. Papperitz, E. "Geometrographie," *Encyklopädie der Mathematischen Wissenschaften* 3(1)4, Geometrie 528–531, 1907.
14. Row, S. *Geometric Exercises In Paper Folding*, Chicago: Open Court Publishing Company, 1905.
15. Shenton, W. F. The First English Euclid, *The American Mathematical Monthly* 35:505–512, Dec. 1928.
16. Shively, L. S. *Modern Geometry*, John Wiley and Sons, 1946.
17. Steiner, J. *Geometric Constructions With A Ruler*, New York: Scripta Mathematica, Yeshiva University, 1950.
18. Tucker, C. O. "The Construction for Mean Proportional," *The Mathematical Gazette*, 14:542–544, 1929.
19. Yates, R. C. *Geometrical Tools*, St. Louis Educational Publishers, 1949.
20. Young, G. C. and Young, W. H. *Der Kleine Geometer*, Berlin, 1908.

Unorthodox Ways to Trisect A Line Segment

Charles W. Trigg

Five methods for trisecting a line segment are offered here in the hope that they may stimulate some of the better geometry students to prove the constructions and to generalize the procedures [illegible]ng.

To provide a means of comparison of the methods, we use a geometrographic index which is the total of the operations performed in the construction. Starting with closed compasses, opening to a particular setting is counted as one operation, as is changing the compasses to another setting. Drawing a straight line, describing a circle or striking an arc are each counted as one operation. In the following procedures, the number of the step is indicated by the number in parentheses. Throughout *(A)* indicates the circle with center at *A*. The dotted lines in the figures are suggestions toward proofs of the constructions.

1. *Modified conventional method.* In order to reduce its geometrographic index, the conventional method has been slightly modified.

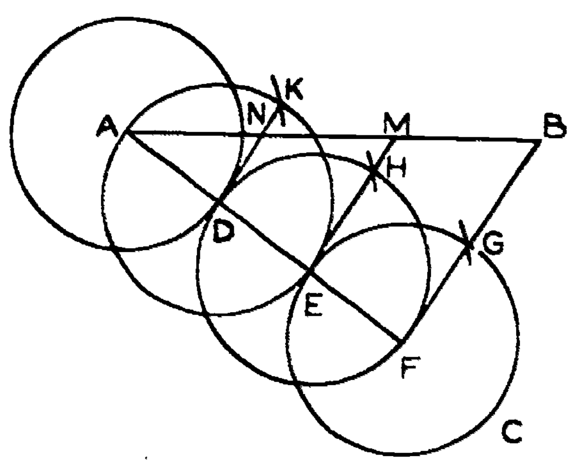


FIGURE 1

Given the line segment *AB*. Through *A* draw line *AC* making an arbitrary acute angle with *AB* (1). Open compasses to an arbitrary radius (2). With *A* as center describe circle cutting *AC* in *D* (3). With *D* as the center describe circle cutting *AC* in *E* (4). With *E* as center describe circle cutting *AC* in *F* (5). With *F* as center describe circle (6). Draw *FB* cutting circle *(F)* in *G* (7). Set compasses to radius equal to *EG* (8). With *D* as center describe arc cutting circle *(E)* in *H* (9). With *A* as center describe arc cutting circle *(D)* in *K* (10). Draw *EH* cutting *AB* in *M* (11). Draw *DK* cutting *AB* in *N* (12). This completes the trisection of *AB*.

2. *Method based on parallel lines.* (Called to the writer's attention by Fred Marer.) Open compasses to an arbitrary radius (1). With *A* and with *B* as centers describe circles cutting *AB* in *C* and *D*, respectively (2) (3). With *C* and *D* as centers describe arcs cutting *(A)* and *(B)*, respectively in points *E* and *F* on opposite sides of *AB* (4) (5). Draw *AE* extended and *BF* extended (6) (7). With *E* and *F* as centers describe arcs cutting *AE* and *BF* in *G* and *H*, respectively (8) (9). Draw *GF* cutting *AB* in *M* (10) and *EH* cutting *AB* in *N* (11), thus completing the trisection.

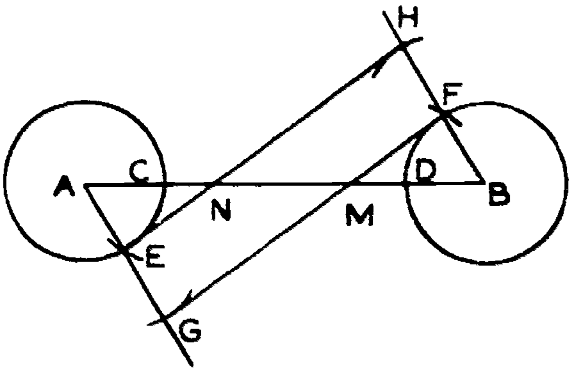


FIGURE 2

3. *Modified method based on parallel lines.* Open compasses to an arbitrary radius (1). With *A* and with *B* as centers describe circles cutting *AB* in *F* and *G*, respectively (2) (3). Through *A* draw *AC* at an arbitrary angle to *AB* and cutting circle *(A)* in *D* (4). With *D* as center describe arc cutting *AC* in *E* (5). Change opening of compasses to radius equal to *DF* (6). With *G* as center describe arc cutting circle *(B)* on the opposite side of *AB* from *E* in *H* (7). Draw EH cutting *AB* in *M* (8). Set compasses to radius equal to *BM* (9). With *M* as center describe arc cutting *AB* in *N*, thus trisecting *AB* (10).

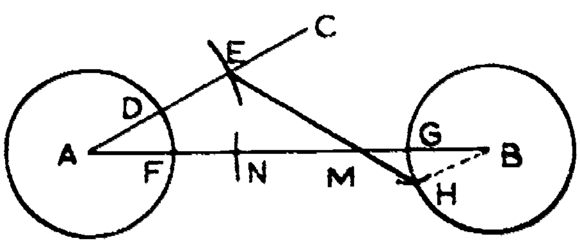


FIGURE 3

4. *Method based on a square.* Open compasses to radius greater than $\frac{1}{2}AB$ (1). With A and B as centers describe arcs intersecting in C and D (2) (3). Draw line through CD cutting AB in E (4). Change opening of compasses to radius equal to AE (5). With E as center describe circle cutting CD in F and L (6). With F as center describe arc cutting CD in G (7).

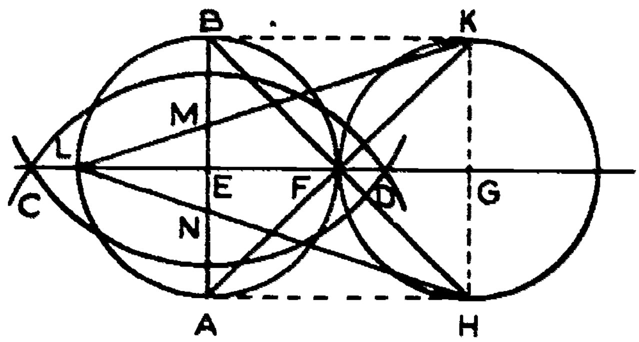


FIGURE 4

With G as center describe circle (8). Draw BF meeting (G) in H (9). Draw AF meeting (G) in K (10). Draw KL and HL cutting AB in M and N, respectively, thus trisecting AB (11) (12).

5. *Method based on Ceva's Theorem.* Extend AB in both directions (1). Open compasses to an arbitrary radius greater than $AB/5$ and less than AB (2). With A and B as centers describe circles cutting BA extended in C, and AB extended in D (3) (4). With C and D as centers describe arcs cutting BA extended in E, and AB extended in F (5) (6).

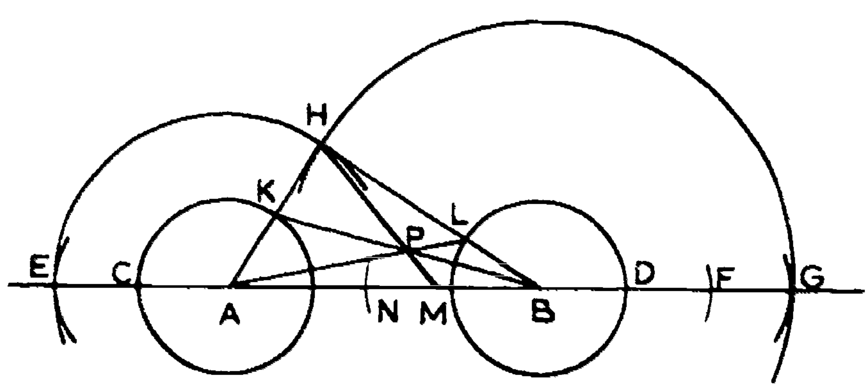


FIGURE 5

With F as center describe arc cutting AB extended in G (7). Change compasses to radius equal to AE and with A as center describe circle (8) (9). Change compasses to radius equal to BG and with B as center describe circle intersecting large circle (A) in H (10) (11). Draw AH meeting small circle *(A)* in K (12). Draw BH meeting small circle *(B)* in L (13). Draw BK and AL meeting in P (14) (15). Draw HP meeting AB in M (16). Change compasses to radius equal to MB and with M as center describe arc cutting AB in N (17) (18).

Generalizations. The values assumed by the geometrographic index when the five methods are generalized to n-secting are given in the body of the following table for various values of n.

Method \ n	2	3	n
1	9	12	$3n+3,\ n>1$
2	6	11	$3n+2,\ n>2$
3	7	10	$2n+4,\ n>2$
4	4	12	$3n+4,\ n>3$
5	13	18	$2n+12, n>2$

The index for $n=2$ in Method 3 may be reduced to 6 by retaining the same radius after drawing *(A)* and *(B)* and then striking arcs with F and G as centers. The join of the intersections of the arcs with *(A)* and *(B)* bisects AB.

In the generalization of Method 4 the square becomes a rectangle.

Care must be taken in Method 5 that the initial opening, r, of the compasses be such that $AB\ /(n+2) < r < AB/(n-2)$.

For $n>2$, Method 3 is always the most efficient one of the five methods. When $n>9$, Method 5 moves up to second place in the order of efficiency.

Selected references for further reading and study

1. Archibald, R. C. Constructions with a double-edged ruler. *Amer. Math. Monthly*, 25:358–360; 1918.
2. Archibald, R. C. Geometrography and other methods of measurements of geometry. *Amer. Math. Monthly*, 27:323–326; 1920.
3. Barnett, I. Geometric constructions arising from simple algebraic identities. *School Science and Mathematics*, 38:521–527; 1938.
4. Bieberbach, Ludwig. Theorie der geometrischen Konstruktionen. Basel: Verlag Birkhauser, 1952. 162 p.
5. Eves, H. and Hoggatt, V. Euclidean constructions with well-defined intersections. *Mathematics Teacher*, 44:262–263; 1951.
6. Fourrey, E. Procédés originaux de constructions géométriques. Paris: Librairie Vuibert, 1924. 142 p.
7. Hobson, E. W. On geometrical constructions by means of the compass. *Mathematical Gazette*, vol. 7, p. 49–54; March 1913.
8. Hudson, Hilda P. [In Squaring the Circle and Other Monographs, New York, Chelsea Publishing Company, 1953.] "Ruler and Compass," 143 p.
9. Lebesgue, Henri. Leçons sur les constructions géométriques. Paris: Gauthier-Villars, 1950. 304 p.
10. Moise, Edwin. In Elementary Geometry from an Advanced Standpoint, Addison-Wesley, 1963. "Constructions with Ruler and Compass," p. 214–241.
11. Petersen, Julius. Méthodes et théories pour la resolution des problèmes de constructions géométriques. Paris: Gauthier-Villars, 1946. 112 p.
12. Rademacher, H. and Toeplitz, O. [In the Enjoyment of Mathematics, Princeton University Press, 1957.] "The Indispensability of the Compass for the Constructions of Elementary Geometry," p. 177–187.
13. Steiner, Jacob. Geometrical Constructions with a Ruler Given a Fixed Circle with Its Center. (Trans. by Marion Stark). New York: Scripta Mathematica, Yeshiva University, 1950. 88 p.
14. Struyk, Adrian. Drawing with a ruler and paper. *School Science and Mathematics*, 45:211–214; 1945.

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